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Feynman's Corner Rule; Quantum Propagation from Special Relativity

G.N. Ord

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Abstract Feynman's sum-over-paths prescription for the Dirac equation in a two dimensional spacetime can be formulated to give an unconventional view of the relationship between quantization and special relativity. By considering a local rule for the maintenance of Lorentz covariance in a discrete space, one is able to see the origin of Feynman's rule and, taking a continuum limit at the last step, one obtains the Dirac propagator as a manifestation of special relativity, rather than a formal addition to it. In this route to the Dirac equation, the path-dependent phase of wavefunctions, relativistic or not, is a direct manifestations of path-dependent proper time.

Keywords Special relativity · Quantum mechanics · Feynman chessboard

1 Introduction

Minkowski spacetime and quantization are pillars upon which much of modern physics is built. The former allows us to transplant classical mechanics into a framework that respects Lorentz covariance. The latter allows us to replace the classical concept of point particle dynamics with wave propagation. Both concepts inherit assumptions of smoothness and scale from Newtonian mechanics, modifying them in different ways.

Dirac successfully married the two concepts mathematically in his famous equation. The physics implicated by the equation is currently interpreted in terms of quantum field theory with a second application of the idea of quantization. Pursuit of a quantum theory of gravity would presumably extend this program to include General Relativity but it appears that a better understanding of the relationship between spacetime and quantum mechanics may be necessary to make progress.

A possibility, minimally explored in this paper, is that both quantum propagation and Minkowski spacetime are manifestations of the same thing. If this is the case then the Dirac equation as usually 'derived' represents a marriage of siblings. Its efficacy results from the

G.N. Ord (✉)
Mathematics Dept., Ryerson University, Toronto, ON, Canada
e-mail: gord@ryerson.ca

fact that Minkowski spacetime and Dirac quantization share a common origin. Its intransigence with respect to interpretation and further generalization may result from the same reason. The common cause needs extension rather than the equation itself.

The Feynman chessboard model,[1] a sum-over-paths route to the Dirac equation, is based on what has become known as 'Feynman's corner rule'. The rule is usually interpreted as a clever trick that invokes the formal analytic continuation necessary to convert a diffusion equation to a wave equation. As a result the model is frequently regarded as an *adaptation* of Feynman's sum-over-paths formulation of quantum mechanics to special relativity rather than the progenitor of the path-integral approach. This note shows that the corner rule actually has its origin in a local implementation of Lorentz covariance. This suggests that, at least in two dimensions, special relativity and quantum mechanics share a common origin that is usually hidden by independent invocations of Minkowski spacetime and quantization.

To emphasize the common origin of Minkowski spacetime and the Dirac equation we shall adjust the two relativity postulates, allowing us the freedom to imitate a background spacetime through fine-scale geometry. Rather than have a picture where "Spacetime tells particles how to move"[2] we create a local rule that guides the particle, imitating the effect of an ambient spacetime. The goal here is to formulate the physics *before* invoking the continuum limit required by a differential description. In the model we discuss there are *considerable* advantages in doing this. The local rule for enforcing Lorentz covariance ultimately removes the formal clothing separating special relativity and quantum propagation. By the time we have explored the usual consequences of special relativity in light of the local rule, we find that quantum propagation is a natural feature that is discovered by paying close attention to path-dependent proper time. In Sect. 2 we state the modified relativity postulates and introduce a hypothetical particle called an 'EAPP' for Euclidean Area Preserving Particle. We discuss the EAPP and how it approximates a conventional particle with a smooth world line.

In the following section we take EAPPs into the realm of stochastic processes, rediscovering Feynman's Chessboard model. Here we see that the existence of the path-dependent proper time, if maintained in the continuum limit, results in the Dirac and Schroedinger equations. In the last section we summarize the advantages and disadvantages of this model and suggest directions for further work.

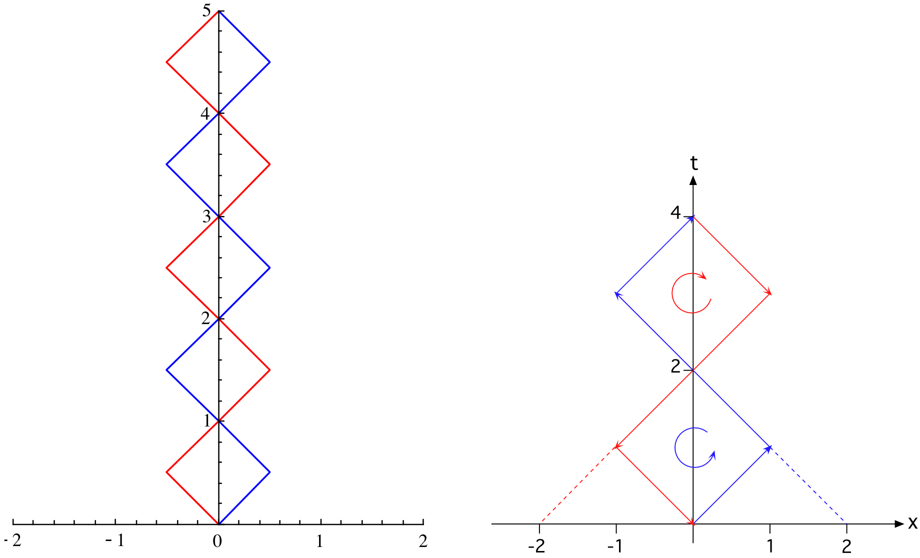
2 Euclidean Area Preserving Particles

In special relativity, the speed of light is a characteristic speed of deep significance. It is the speed of photons in free space, but it is also 'known' to massive particles through the famous relation to energy $E = mc^2$. In most expositions of special relativity this fact comes out when one considers conservation of momentum and energy in light of the Lorentz transformation [3–5].

The conventional concept of the world-line of a free particle is not itself imbued with any information regarding a particle's mass. Mass is simply a background attribute assigned to a world-line so that it can correctly model the behaviour of a real particle in connection with

[1] In the interest of simplicity, the chessboard model is often formulated in the absence of a coupling to the electromagnetic field. This may be included [1], however, since our interest is primarily the corner rule, we restrict ourselves to the free particle case.

[2] "Spacetime tells matter how to move. Matter tells spacetime how to curve." One of J.A. Wheeler's aphorisms describing general relativity [2].



(a) A chain of spacetime areas.

(b) The basic repeating unit.

Fig. 1 (**a**) A particle at rest as a chain of spacetime areas. Here $c = 1$ and the particle, from the point of view of measuring its instantaneous speed, can be identified with the right-hand boundary. The sequence of crossing points and a smooth interpolant between them is the EAPP equivalent of a world line. (**b**) Two links in a chain of oriented areas. The orientation arises from a peculiarity of the Minkowski metric that for example links the $(x, t) = (1, 1)$ to the $(-1, 3)$ event

other particles and forces. However, this picture ultimately clashes with that of quantum propagation since mass sets the frequency scales of the quantum wave equations. As a result we shall depart from standard approaches and modify the picture behind world-lines. The idea is that there should be a simple fine-scale feature of the worldline that distinguishes a particle's mass. The feature has to be fine-scale so that on large scales we can revert to the picture of a smooth worldline informed by an ambient spacetime.

We can do this in a simple way by exploiting Nature's universal recognition of the speed of light. We shall require particle worldlines to have instantaneous speeds $\pm c$ almost everywhere. Average speeds $v < c$ are then generated by employing fine-scale motion that preserves an intrinsic area. As variants of the usual relativity postulates we propose:

1. The laws of physics are identical in all inertial frames on large scales.
2. All material particles move with speed c almost everywhere.

We have weakened the first postulate to allow flexibility with regard to scale. The strengthened second postulate forces a fine scale motion on massive particles. In a two dimensional spacetime the postulates force particle trajectories to have a zig-zag appearance as in Fig. 1(a).

For reasons that will become clear, we shall think of such paths as occurring in pairs that form a chain of oriented spacetime areas. The orientation of these areas is a local mechanism for enforcing the Minkowski metric that would appear in a conventional approach and must appear here on large scales. In the Minkowski metric, spacetime events that may be connected by a light-like path are equivalent in the sense that the metric distance between them is zero. In Fig. 1(b) we see that the spacetime point $(x, t) = (1, 1)$ is on the same null geodesic as the point $(x, t) = (-1, 3)$. It is as if the space coordinates are interchanged from one area to the other. This feature is accounted for in an EAPP by orienting the areas of the two successive links. If the lower area is oriented positively according to the right-hand rule, the upper area is oriented negatively by the same rule if we consider the blue path from $(0,0)$ to $(0,4)$ to be directed as in the figure with the red path inheriting the appropriate direction to orient the two areas.

By analogy with worldlines we call the sequence of areas *world-chains*, the figure-of-eight pictured in Fig. 1(b) providing the basic repeated unit. The EAPP pictured in Fig. 1 can be thought of as an approximation to a massive particle at rest. The ubiquitous presence of c is facilitated by the fact that each link in the chain has boundaries with slope $\pm c$. The 'at rest' feature is a manifestation of the fact that if we construct a linear interpolation of the crossing points of the chain, the result is the conventional world line of a particle at rest. The qualification of 'large scales' in our first postulate means scales much larger than the distance between crossing points of the chain. To give an idea of scale in the figure, if the EAPP is to mimic an electron, the time interval between crossing points is of the order of 10^{-22} seconds and the 'width' of the chain is of the order of 10^{-12} centimetres, both scales well below the effective limits of the classical behaviour of the electron. For simplicity we employ the same unit of measurement for both space and time, absorbing c into the t variable so light-like paths have slope ± 1 on spacetime diagrams and massive particles have average velocities $-1 < v < 1$.

We want the EAPP to respect Lorentz covariance and it is not too difficult to see how we can do this using Euclidian areas. Let us assume that if we observe the EAPP in Fig. 1 from an inertial frame moving with respect to the lab frame with velocity $-v$, it will look just like a particle moving with velocity $+v$ on the same spacetime diagram. The fact that an EAPP has velocity $+v$ means that the interpolant connecting the chain of crossing points will be a straight line with slope $\triangle x/\triangle t = v$. To be in accord with the second postulate the boundaries of the links must still have slope $\pm c$. Furthermore, the first crossing point that was located at, say $(x,t) = (0,t_0)$ in the lab frame must be mapped onto $(x,t) = (x',t')$ where $t'^2 - x'^2 = t_0^2$.

To see this, note from the right side of Fig. 4 that the area of the first link in the lab frame is

$$A_0 = t_0^2/2. \tag{1}$$

The area of the same link in the moving frame is

$$A' = 2t_R t_L, \quad \text{where } t_L = t' - t_R. \tag{2}$$

If we require the areas to be the same in both frames then $A' = A_0$ so

$$t_0^2 = 4t_R t_L. \tag{3}$$

Now

$$\begin{aligned} t' &= t_R + t_L \\ x' &= t_R - t_L = vt' \end{aligned} \tag{4}$$

so

$$\begin{aligned} t_R &= \frac{1}{2}(t' + x') \\ t_L &= \frac{1}{2}(t' - x'). \end{aligned} \tag{5}$$

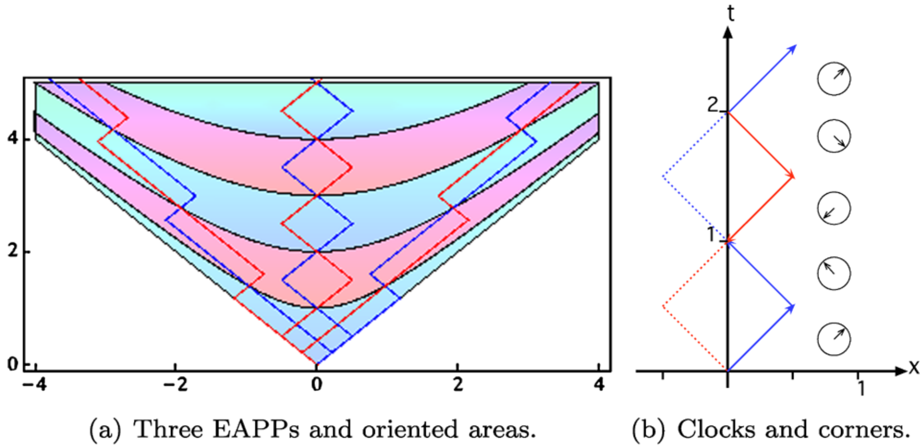


Fig. 2 (**a**) A few EAPPs aligned at the origin. The ensemble of these world-chains representing free particles partitions spacetime into areas of opposite orientation. (**b**) The enumerative paths of an EAPP are such that the oriented paths provide a 'digital watch' that ticks with each corner in the path. This provides us with a convenient way of measuring proper time using complex numbers. In the figure is the right enumerative path of an EAPP. On the right of each link is the digital watch that ticks at each corner, keeping parallel to the oriented path. The association with complex numbers is that if the watch hand is a unimodular complex number, each tick represents multiplication by i

Combining (3), (4) and (5) we get

$$t' = \frac{t_0}{\sqrt{1 - v^2}}. \tag{6}$$

Thus the requirement that the EAPP area be invariant under a change of inertial frames forces the appropriate Lorentz transformation. To allow arbitrarily small velocity changes, the orientation of the area must also be preserved. A sketch of a few EAPPs with the resulting oriented areas appears in Fig. 2(a).

The ensemble of crossing points provides a sketch of a grid in Minkowski space. For example the locus of points of first crossings of all our particles of arbitrary v consists of the locus of all points whose time-like distance from the origin Fig. 2. Our world-chains accommodate the Lorentz transformation through the imposition of the local and frame independent requirement that the Euclidean area of our chains be preserved. The ensemble of EAPPs also partitions spacetime into regions of positive and negative orientation, a feature that is not apparent in the smooth-worldline paradigm! The extra feature of orientation of areas means that the entire ensemble of free particle paths agree on the orientation of spacetime areas within the future light cone of the origin. This is illustrated in Fig. 2(a) through the alternate shading of areas of positive and negative orientation.

The crossing points and corners of the EAPPs can be thought of as the ticks of an intrinsic clock carried by the particle. In Fig. 2(b) we see that the basic repeating unit of two areas has four possible paths from bottom to top corresponding to two possibilities for the bottom two links and two possibilities for the top two. The two paths that maintain colour and direction in the figure have two corners. The two outer boundaries that change colour at the crossing point each contain three corners. The principal of maximal ageing suggests that these two, of the four possible, should count the proper time of the particle. We call these paths ‘enumerative’ for their role in counting proper time and they form our analog of ‘world-line’. The right hand boundary of a chain has links with directions that rotate counterclockwise with period four. This useful feature can be used as a clock.

The proper time of an EAPP differs from the usual proper time of special relativity in that it is digital. To distinguish it from conventional proper time we shall borrow the expression wristwatch time [5] to remind us that it is carried with the particle. As we shall see, the interval between ticks is determined by the mass of the particle, the clock itself being encoded in the spacetime geometry of the enumerative path. To see this we look at the twin paradox for EAPPs since it illustrates an important feature that does not appear explicitly in the usual formulation of special relativity.

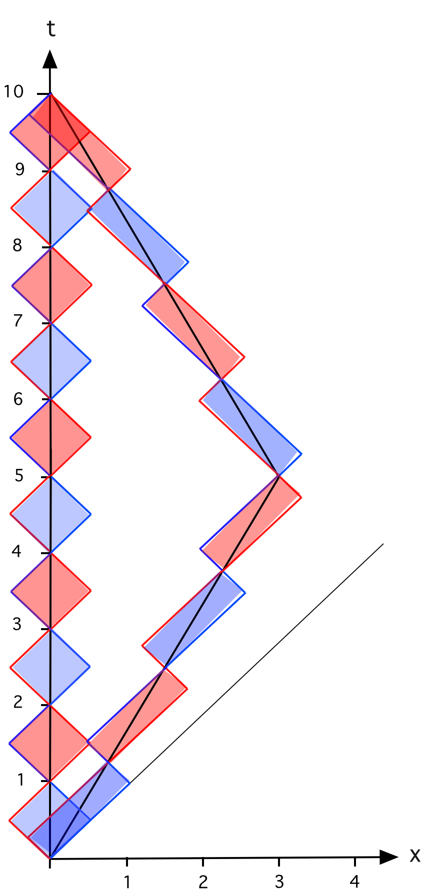
In Fig. 3(a) two chains are compared. The chain representing the inertial twin is at rest in the lab frame, the rocket twin moves at speed $v = 3/5$ out to the point $x = 3$ and back to the origin. As expected the wristwatch time of the rocket twin is 8 compared to 10 in the rest frame. This is evident by just counting time as units of area in the two chains. A feature that will be important later can be seen in the figure. Where the world lines cross at $t = 10$, the chains have an overlapping area. The areas of overlap have the same orientation. However orientation is clearly *path dependent* and paths that cross may intersect with areas of opposite orientation. This is the precursor of phase in this model. Any two non-identical paths between two points in spacetime will in general give different wristwatch times and different orientations at the end point.

EAPPs clearly have an internal structure that digitally counts their wristwatch time. This is a feature that is very useful and worth exploring. It lies at the heart of the ubiquitous presence of complex numbers in quantum propagation, and the odd signature of spacetime in this model.

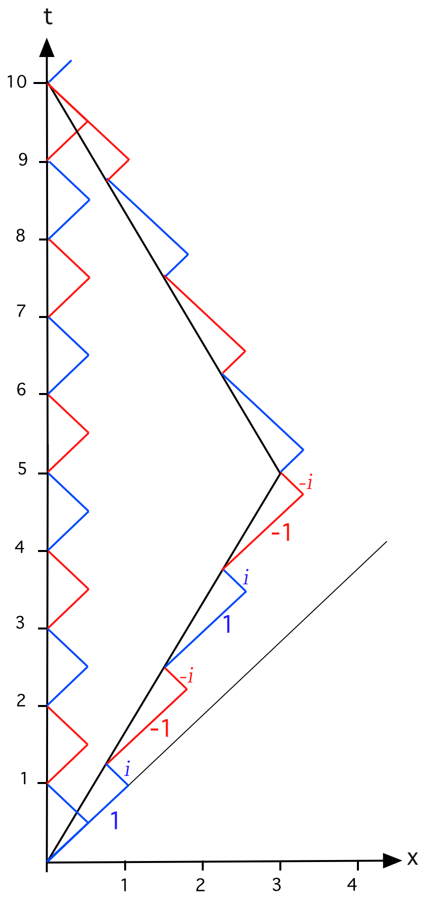
Along a world chain, the wristwatch ticks at each corner of the right-hand boundary, and a complex number may be used to represent a vector that stays parallel to the oriented right-hand boundary of the EAPP Fig. 2(b). The association of complex numbers with the four directions of oriented areas allows us to associate i with every tick of the clock (Fig. 3(b)). That is, if our digital watch is a unimodular complex number, multiplication by i effects a ‘tick’. A tick corresponds to multiplication by i. The argument of $e^{i\theta}$ counts the ticks in units of π. Real or imaginary determines right or left moving in the enumerative path.

This is illustrated in Fig. 3(b) for the first part of the rocket twin path where each link is assigned one of the fourth roots of unity. Let us use the unit imaginary and the corners in the paths to count time for the twins. For the inertial twin the right enumerative path from $(x, t) = (0^+, 0^+)$ to $(0^+, 10^+)$ has 20 corners resulting in i^{20} for a proper time of 10 ($i^{20} = e^{10\pi i}$), an orientation of $+1$ ($i^{20} = 1$) and a final enumerative direction along the right light cone (i^{20} is real and positive). Similarly the rocket twin has a path with 16 corners for a proper time of 8 with a final orientation as for the inertial twin. The ultimate reason that complex numbers are implicated in the counting process here is the fact that we are dealing with oriented *areas* rather than the smooth curves of conventional world-lines. The oriented areas have orientation ± 1 so *the counting process involved in the statistical mechanics involves a periodic use of subtraction as well as addition*. The translation of this counting from area to length through a square root then invokes the unit imaginary. The rule itself (“associate i with every corner in the path”) is Feynman’s corner rule and will reappear later in association with his Chessboard Model [6].

Thus far EAPPs fulfill the kinematic requirements of special relativity on scales greater than the chainlink size. The area preserved is the product of the projections of the enclosed area onto the light-cone boundaries. For free particles, the crossing-point ticks are determined by two fixed frequencies, one on each of the cones. An EAPP always sees these two



(a) The Twin Paradox.



(b) Digital Time.

Fig. 3 (**a**) The 'Twin Paradox' with a reflected EAPP. A chain of oriented areas 'at rest' at the origin represents the inertial twin in its rest frame. The rocket twin moving at speed $3/5$ the speed of light is represented by a chain that is reflected back at the spacetime point $(x, t) = (3, 5)$ in the lab frame. The moving EAPP's wrist watch time is 8 compared to the inertial time of 10. (**b**) Counting corners in the twin paradox example. Associating the unit imaginary with every corner in a path keeps track of the number of ticks of the clock as the argument of the exponential $e^{i\theta}$. As we progress along the right enumerative path of the rocket twin the digital clock ticks at the corners as $\{e^0, e^{i\pi/2}, e^{3i\pi/2}, \ldots\}$. The rule of i for every corner of the path will appear later when we consider the Dirac equation

frequencies as equal on his wristwatch. An observer in a lab frame moving with respect to the EAPP will see two different frequencies (Fig. 4). The moving observer needs both of these frequencies to track the motion of the EAPP in his reference frame. The two frequencies correspond to digital signals that oscillate independently along the light cones between plus and minus one. Comparing these oscillations to simple harmonic oscillators we would expect energies proportional to the frequencies so we write

$$E = h(\nu_R + \nu_L), \tag{7}$$

where h is some fixed positive constant and the ν are the frequencies on the two light cones.

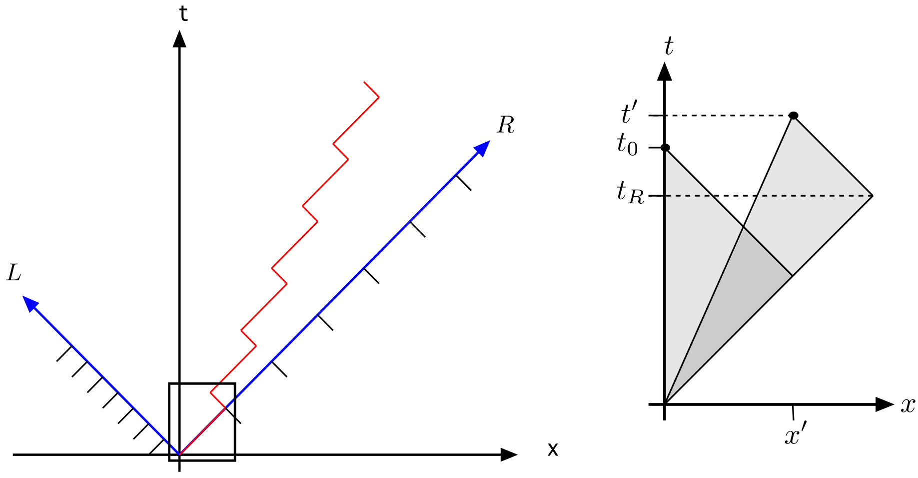


Fig. 4 The right-enumerative path of an EAPP moving at constant speed. The frequency of direction changes on the left and right light cones depends on the macroscopic velocity of the particle. If the particle is at rest the two frequencies are the same. If the particle is moving the two frequencies differ. The right part of the figure expands the first half link, comparing it to the same link in the frame of the particle. The two frequencies ν_R and ν_L have corresponding wavelengths t_R and $t_L = t' - t_R$ respectively

As frequencies, ν_r and ν_l are just inversely proportional to the lengths of the zigs and zags. Referring to Fig. 4 and (4), (5) and (6) we have:

$$\nu = \frac{1}{t_R} + \frac{1}{t_L} = \frac{4t'}{t'^2(1-v^2)} = \frac{\nu_0}{\sqrt{1-v^2}} \equiv \gamma\nu_0. \tag{8}$$

Our proposed energy is then

$$E = h\gamma\nu_0 \approx h\nu_0 + \frac{1}{2}h\nu_0 v^2 \tag{9}$$

the latter being the case in the event that $v \ll 1$. If we identify the EAPP rest mass with $m = h\nu_0$, (8) gives the correct velocity dependence of the relativistic mass. Similarly if we write $p = m\gamma v$ we get the required $E^2 = m^2 + p^2$.

These arguments show that the oriented area construction is sufficient to have EAPPs behave on large scales as if they were massive particles moving in Minkowski space. In the next section we consider more closely the effect of orientation on fine scales.

3 Sum Over Paths

When discussing the twin paradox of Fig. 3(a) we noted that orientation is *path-dependent* and is a function of the particle's wristwatch time. From Fig. 2(b) it is apparent that a convenient measure of orientation is a complex number that gives us a digital readout of the path's wristwatch time. From Fig. 3(b) it is clear that along any given path, starting at the origin, the orientation at the end of the path will be i^R where R is both the number of corners in the path and the number of ticks of the particle's wristwatch.

Let us now introduce a stochastic element. Consider a lattice with spacing $\epsilon \ll t_0$ where t_0 is the first crossing point of our free particle EAPP. We generate a stochastic EAPP in the following way. We always step along diagonals on the lattice in the positive t direction. At each step we usually maintain our current direction but occasionally switch direction introducing a corner in the path, with probability $\epsilon m \ll 1$. Such paths look just like our enumerative paths except the individual links ultimately have lengths governed by the exponential distribution. Now consider the following sum:

$$K(b,a,\epsilon) = \sum_R N(R)(i\epsilon m)^R, \tag{10}$$

where b is a positive timelike distance from a. The sum here is over all the stochastic 'Chessboard-like' paths between a and b. The paths are partitioned with respect to the number of corners R in each path. In terms of EAPPs, R is the wristwatch time along the path, i^R gives the orientation at the end of the path and $(\epsilon m)^R$ is proportional to the probability that a particular path has R corners. Clearly, all R-paths have the same wristwatch time and consequently the same digital watch state. There are only four such states, but the sum over R will mix them as linear combinations of the four complex numbers. The result will be another complex number that will interpolate between the original set of four watch-ticks. Equation (10) simply calculates an expected value of the orientation over all possible lattice paths, the variation of orientation being a result of path dependent proper time!

The sum in (10) is in fact well known. It is the same sum as the Chessboard model due to Feynman.[3] In the limit as $\epsilon \to 0$ it approaches the propagator for the Dirac Equation [7]. The formulation in (10) is Feynman's version of the sum-over-paths for a relativistic particle in two dimensions.

Formula (10) is usually the starting point for a demonstration of the path integral formulation for the Dirac Equation [7–9], the relation to Kac's model of the telegraph equations through analytic continuation [10], the interpretation of world chains as a single path [11–13] or an exploration and development of discrete physics [14]. It is a formula with a small but noticeable place in the history of quantum mechanics if for no other reason that it provided a guide to Feynman's thinking [15]. In all these contexts Feynman's rule of 'i for every corner of the path' appears as a feature that eventually ties the model into the Dirac equation. The difference here is that we have arrived at (10) as a stochastic variant of a model that requires massive particles to preserve Lorentz covariance through local geometry. After the continuum limit the sum has a conventional interpretation as a formula for a free particle in Minkowski space after quantization. Prior to the continuum limit the paths themselves and the method of counting are nothing more than a stochastic version of EAPPs. The local rule for preserving Lorentz covariance for a massive particle has done more than initially requested, it has given us Lorentz covariance *and* quantum propagation.

Two for the price of one is economical, but it also suggests the possibility that attempted marriages of relativity and quantum mechanics may miss-interpret a feature that they share. For example, consider forming the sum in (10) in the limit as $\epsilon \to 0$. We get, in the non-relativistic approximation $v \ll 1$, in conventional units, [6].

$$K(b,a) = \exp[-imc^2(t_b - t_a)/\hbar]\left(\left(\frac{2\pi i\hbar(t_b - t_a)}{m}\right)^{-1/2} \exp\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right) \tag{11}$$

We can now read this formula simply in terms of EAPPS. The product of the exponentials is just the expected orientation based on wristwatch time over the ensemble of paths using the approximation $(1 - v^2)^{1/2} \approx 1 - v^2/2$. In the conventional picture the formula is

[3] Reference [6] problem 2.6.

just Feynman's non-relativistic kernel multiplied by a very rapidly varying rest mass term that acts like a carrier wave. When we remove the rest mass term, the remainder obeys the Schroedinger equation. From the perspective of EAPPS, Schroedinger's equation acquires its form as a diffusion equation with an imaginary diffusion constant as an inheritance from Lorentz covariance and the resulting path dependent proper time! We do not find c explicitly in the Schroedinger equation simply because it drops out in the first order term in the small v expansion of $m\gamma c^2$. However *the inheritance of phase from proper time is clear and unequivocal*.

From the EAPP point of view, the non-relativistic path integral also makes perfect sense. The usual Feynman kernel, in brackets in (11) is essentially the usual Gaussian kernel of the Wiener process that one would get from the Kac model of diffusion [16, 17], except the corner weight of 1 in the Kac model is replaced by i *to count the proper time of the path*. The path-dependent phase of Feynman paths in the non-relativistic path integral implements the path-dependent proper time of EAPPS in the non-relativistic approximation!

4 Discussion

The Chessboard model was developed by Feynman in a period when he was trying to understand the Dirac equation from as many points of view as possible. Regarding this process he commented to his friend T.A. Welton [18]:

> The power of mathematics is terrifying—and too many physicists, finding they have correct equations without understanding them, have been so terrified they give up trying to understand them. I want to go back & try to understand them. What do I mean by understanding? Nothing deep or accurate—just to be able to see some of the qualitative consequences of the equations by some method other than solving them in detail.

This article takes this view with the Chessboard model itself. In conventional approaches to relativistic quantum mechanics one inherits a picture of Minkowski spacetime from classical physics. This picture for free particles is *scale independent* and assumes that worldlines are smooth and free of mass-dependent geometry. One modifies the picture to encompass wave propagation by replacing dynamical variables by differential operators. The prescription works for the Dirac equation, but it is a marriage of two different pictures, both based on the idealization of a smooth featureless continuum.

By comparison, the EAPP route to the Dirac equation in two dimensions is transparent and displays an intimate relationship between special relativity and quantum propagation. By building fine-scale mass-dependent geometry into worldlines by a local rule that preserves a Euclidean area associated with the worldline, we imitated the behaviour of a classical particle embedded in Minkowski spacetime on coarse scales. In doing so we discovered Feynman's corner rule for counting proper time and its connection to oriented areas. By examining the twin paradox we noticed that path-dependent proper time implied path-dependent orientation. A sum-over-paths of a particle's wristwatch time was then immediately recognizable as the chessboard model.

There are several interesting features in this route to the Dirac equation.

– Mimicking Minkowski space through preservation of spacetime area builds the feature (oriented area) that manifests itself in quantum propagation. An invocation of Minkowski space at the beginning of the calculation would have hidden this feature beneath a smoothness assumption.

– The path-dependent phase of wavefunctions is a manifestation of path-dependent proper time. This is the case for both the Dirac and Schroedinger equations.
– The odd signature of spacetime in special relativity and the ubiquitous implication of complex numbers in non-relativistic quantum mechanics both arise from the same source. The source is the *signed* area metric that is built into an EAPP as a local rule, namely:

$$(\Delta s)^2 = \pm((\Delta t)^2 - (\Delta x)^2) \tag{12}$$

On large scales this is the source of the odd signature of the conventional Minkowski metric where one chooses the sign according to whether events are spacelike or timelike separated. *On small scales the EAPP construction uses null links implicating both signs in the area metric and the unit imaginary in the length metric.*

Given the restricted context of this model to a two dimensional spacetime, there can be two contrasting speculations about the generality of this picture.

1. The EAPP picture is an artifact of two dimensions made possible by the fact that there are only four possible directions in spacetime. In a four dimensional spacetime the simplicity of only four directions is lost, making the situation more complicated and removing the common connection. This is reflected in the lack of any consensus on an extension of the Chessboard model to four dimensions.
2. The EAPP picture is likely to be more general than the 2D model. The arguments that motivate the model do not depend on dimension. The result of the model is that the path dependence of the proper time of special relativity manifests itself as the path dependent phase of quantum mechanics. Since proper time itself is a Lorentz scalar the result may be expected to carry over to four dimensions by a simple embedding.

We cannot resolve these contrasting possibilities here although the author favours the second conclusion and will publish an extension of the Chessboard model to four dimensions featuring oriented areas in due course.

There is a fairly extensive literature on Zittebewegung in the Dirac equation with opinions on the phenomenon varying from considering it an artifact and a distraction, to being a phenomenon central to quantum mechanics [19]. This model sides strongly with the latter view in that the EAPP is a specific model of Zitterbewegung in two dimensions.

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